

Vector-Tensor Duality

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Abstract

A dynamical non-abelian two-form potential gives masses to vector bosons via a topological coupling [1]. Unlike in the abelian case, the two-form cannot be dualized to Goldstone bosons. Duality is restored by coupling a flat connection to the theory in a particular way, and the new action is then dualized to a non-linear sigma model. The presence of the flat connection is crucial, which saves the original mechanism of Higgs-free topological mass generation from being dualized to a sigma model.

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The properties of an abelian two-form are well-known. By itself, it describes a massless particle [2], while when coupled to an abelian gauge field via a topological term it provides a gauge invariant mass to the vector [3–6] without a residual scalar Higgs. The abelian mechanism was generalized to a compact non-abelian gauge group sometime ago in the context of non-abelian quantum hair on black holes [7] and then as a way of giving masses to vector bosons [1]. The non-abelian model acquires significance when we consider the fact that the Higgs particle has not been observed yet. If it remains elusive in the coming generation of accelerators, new ways of generating vector boson masses will have to be considered. The non-abelian two-form provides a plausible model for Higgs-free mass generation.

The action for the dynamical non-abelian two-form is

$$S = \int d^4x \text{Tr} \left(-\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m}{4} \epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} F_{\rho\lambda} \right). \quad (1)$$

Here D_μ is the connection of some gauge group $SU(N)$, $H_{\mu\nu\lambda} = D_{[\mu} B_{\nu\lambda]} - [F_{\mu\nu}, C_\lambda]$ is the compensated field strength of a non-abelian two-form B which lives in the adjoint representation, C is an auxiliary field which also lives in the adjoint representation, and $F_{\mu\nu} = [D_\mu, D_\nu]$ is the field strength of the $SU(N)$ gauge field A . The presence of the compensating field C is required [1,8–10] in order to generalize the Kalb-Ramond transformation [11] to a non-abelian one,

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + D_{[\mu} \Lambda_{\nu]}; \quad C_\mu \rightarrow C_\mu + \Lambda_\mu. \quad (2)$$

This symmetry of the action ensures that classically there are only three degrees of freedom for each gauge index. These can be thought of as the three degrees of a massive gauge field, and there is no degree of freedom corresponding to a residual Higgs field. This model can be quantized, and a pole appears in the propagator of the gauge field when tree level diagrams are summed [1,9], and there is no residual Higgs particle. A BRST-invariant quantum action for this model has also been found recently [9,13], and it seems possible that this model can be shown to be renormalizable and unitary.

Any alternative for the Higgs mechanism must be shown to be unitary and renormalizable in order for it to be given serious consideration. There are good reasons to think that the

model is renormalizable – it is by power counting, and the propagators fall off as $1/k^2$. On the other hand, unitarity is less tractable in this model. A calculation of unitarity in scattering of longitudinal vectors from longitudinal vectors runs into problems immediately because the longitudinal mode in (1) is not easily identifiable. In the abelian model one can write a duality relation of the form $H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\rho}(mA^\rho + \partial^\rho\phi)$ where ϕ is a scalar. This allows one to rewrite the action in an obviously unitary form and it can be shown that unphysical modes do not propagate. But no such duality relation exists for the non-abelian model (1). In this letter, I explore a modified version of this model. I shall show that when a flat connection couples to the model in a particular way in addition to the dynamical gauge field already present, the non-abelian antisymmetric tensor can be dualized to a non-abelian version of the abelian duality relation.

To begin with, let me include an $SU(N)$ flat connection \tilde{A} in the model in addition to the $SU(N)$ gauge field A , and define two vector fields \mathcal{A}_μ and Φ_μ ,

$$\mathcal{A}_\mu = \frac{1}{2}(A_\mu + \tilde{A}_\mu), \quad \Phi_\mu = \frac{1}{2}(A_\mu - \tilde{A}_\mu). \quad (3)$$

Φ_μ transforms homogeneously under $SU(N)$ gauge transformations, while \mathcal{A}_μ transforms like a connection. This allows the following covariant derivatives and field strengths to be defined,

$$D_\mu = \partial_\mu + A_\mu; \quad \mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu; \quad F_{\mu\nu} = [D_\mu, D_\nu]; \quad \text{and} \quad \widetilde{H}_{\mu\nu\lambda} = \mathcal{D}_{[\mu}B_{\nu\lambda]}. \quad (4)$$

For the sake of simplicity I have ignored the auxiliary field (in other words set it to zero using the vector gauge transformation) in the definition of \widetilde{H} , but it can be restored without any problem. Now I can write down a modified version of the action (1),

$$S = \int d^4x \text{Tr} \left(-\frac{1}{12} \widetilde{H}_{\mu\nu\lambda} \widetilde{H}^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m}{4} \epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} F_{\rho\lambda} \right). \quad (5)$$

The vector-tensor duality shows up in the dynamics of this action. The equations of motion of the two dynamical fields A_μ and $B_{\mu\nu}$ as derived from this action are

$$D_\nu F^{\nu\mu} - \frac{1}{4} [B_{\nu\lambda}, \widetilde{H}^{\mu\nu\lambda}] + \frac{m}{6} \epsilon^{\mu\nu\rho\lambda} D_{[\nu} B_{\rho\lambda]} = 0, \quad (6)$$

$$\mathcal{D}_\lambda \widetilde{H}^{\mu\nu\lambda} + \frac{m}{2} \epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda} = 0. \quad (7)$$

The second equation of this set is reduced to an identity by the ansatz

$$\widetilde{H}^{\mu\nu\lambda} = -2m\epsilon^{\mu\nu\lambda\rho}\Phi_\rho. \quad (8)$$

Since \tilde{A} is a flat connection, $[\partial_\lambda + \tilde{A}_\lambda, \partial_\rho + \tilde{A}_\rho] = 0$, it follows that

$$\begin{aligned} \mathcal{D}_\lambda \widetilde{H}^{\mu\nu\lambda} &= -m\epsilon^{\mu\nu\lambda\rho}(\mathcal{D}_\lambda \Phi_\rho - \mathcal{D}_\rho \Phi_\lambda) \\ &= -\frac{m}{2}\epsilon^{\mu\nu\lambda\rho} \left(F_{\lambda\rho} - [\partial_\lambda + \tilde{A}_\lambda, \partial_\rho + \tilde{A}_\rho] \right) \\ &= -\frac{m}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}. \end{aligned} \quad (9)$$

So this ansatz solves the equation of motion (7) for $B_{\mu\nu}$. Although the fields appearing in this ansatz were already present in the theory, this is not a constraint. In fact, this ansatz solves the Gauss' Law type constraint $\mathcal{D}_j \Pi_{ij} + \frac{m}{2}\epsilon_{ijk}F_{jk} \approx 0$. So one can quantize the theory after eliminating this constraint by using the ansatz¹. However, this paper will be limited to a classical analysis. This ansatz also simplifies the other equation of motion considerably, as can be seen by rewriting the equation,

$$\begin{aligned} D_\nu F^{\nu\mu} - \frac{1}{4}[B_{\nu\lambda}, \widetilde{H}^{\mu\nu\lambda}] + \frac{m}{6}\epsilon^{\mu\nu\rho\lambda}\widetilde{H}_{\nu\rho\lambda} + \frac{m}{6}\epsilon^{\mu\nu\rho\lambda}[\Phi_{[\nu}, B_{\rho\lambda]}] &= \\ D_\nu F^{\nu\mu} + \frac{m}{2}\epsilon^{\mu\nu\rho\lambda}[B_{\nu\lambda}, \Phi_\rho] - \frac{m^2}{3}\epsilon^{\mu\nu\rho\lambda}\epsilon_{\nu\rho\lambda\tau}\Phi^\tau + \frac{m}{2}\epsilon^{\mu\nu\rho\lambda}[\Phi_\nu, B_{\rho\lambda}] &= \\ D_\nu F^{\nu\mu} - 2m^2\Phi^\mu = D_\nu F^{\nu\mu} - m^2A^\mu + m^2\tilde{A}^\mu &= 0, \end{aligned} \quad (10)$$

which shows clearly that, at least at the classical level, the action (5) describes a massive vector field. This gauge-invariant mass of the vector field can be obtained by rewriting the action by substituting the duality ansatz (8) into it. It is possible to rewrite the $B \wedge F$ interaction term using the flatness of \tilde{A} ,

¹For a given connection A_μ , *any* flat connection \tilde{A}_μ will satisfy $(D + \tilde{D})_{[\mu}(A - \tilde{A})_{\nu]} = \frac{1}{2}F_{\mu\nu}$. It follows that if $\Pi_{ij} = \epsilon_{ijk}(-2m\Phi_k + K_k)$ is the general solution of the constraint equation, K_k as a function of \tilde{A} must satisfy $\mathcal{D}_{[j}K_{k]} = 0$ identically for all flat connections \tilde{A} . It can be shown that such a K_i must vanish.

$$\begin{aligned}
\frac{m}{4} \int \text{Tr} (\epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} F_{\rho\lambda}) &= m \int \text{Tr} (\epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} \mathcal{D}_\rho \Phi_\lambda) \\
&= -m \int \text{Tr} (\epsilon^{\mu\nu\rho\lambda} (\mathcal{D}_\rho B_{\mu\nu}) \Phi_\lambda) \\
&= -\frac{m}{3} \int \text{Tr} (\epsilon^{\mu\nu\rho\lambda} \widetilde{H}_{\mu\nu\rho} \Phi_\lambda) \\
&= -4m^2 \int \text{Tr} (\Phi_\mu \Phi^\mu)
\end{aligned} \tag{11}$$

where I have neglected surface terms in the second line². It should also be noted that $\widetilde{H}_{\mu\nu\lambda} \widetilde{H}^{\mu\nu\lambda} = -24m^2 \Phi_\mu \Phi^\mu$. When all this is put together, the action (5) can be written as

$$S = \int d^4x \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2m^2 \Phi_\mu \Phi^\mu \right). \tag{12}$$

As can be seen, all references to $B_{\mu\nu}$ has dropped out of the action. This action reproduces the equation of motion (10) for a massive A_μ .

Let me make a couple of digressions here. As mentioned before, the duality exists with minor modifications when the compensating auxiliary field C is introduced as in [1]. One has to replace the field strength \widetilde{H} by a compensated field strength \widetilde{H}' ,

$$\widetilde{H}'_{\mu\nu\lambda} = \widetilde{H}_{\mu\nu\lambda} - [F_{[\mu\nu}, C_{\lambda]}] + [\Phi_{[\mu}, D_\nu C_{\lambda]}], \tag{13}$$

and replace \widetilde{H} by \widetilde{H}' in the action, while leaving the other terms as they were. Then the non-abelian vector gauge symmetry (2) is a symmetry of the action. It is easy to see that the rest of the analysis above holds with the replacement of \widetilde{H} by \widetilde{H}' . Therefore, what has been done in this paper boils down to a proof of classical equivalence of a Stückelberg-type theory with a particular theory of dynamical non-abelian two-forms.

A clarification is needed on the issue of enforcing the flatness of \tilde{A} in the path integral. The action of Eq. (5) as written uses $\tilde{F}_{\mu\nu} \equiv [\partial_\mu + \tilde{A}_\mu, \partial_\nu + \tilde{A}_\nu] = 0$, which shows up in Eq. (9). In the path integral this will be implemented by $\delta(\tilde{F}_{\mu\nu})$, which can be rewritten as a

²The surface term is of the form $\int dS^\mu \epsilon^{\mu\nu\rho\lambda} B_{\nu\rho} \Phi_\lambda$. This may have a non-zero contribution in the presence of strings or monopoles if B also carries a topological charge. In such a situation one also has to be careful about substituting the ansatz into the action.

term $\text{Tr}(\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}E_{\mu\nu}\tilde{F}_{\rho\lambda})$ in the action, where $E_{\mu\nu}$ is a *new* Lie-algebra valued two-form. Let me also include the term $\text{Tr}(-\frac{m}{4}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\tilde{F}_{\rho\lambda})$ which corresponds to what was set to zero in Eq.(9). Then the total action is

$$S = \int d^4x \text{Tr} \left[-\frac{1}{12}\tilde{H}_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{4}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}(F_{\rho\lambda} - \tilde{F}_{\rho\lambda}) + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}E_{\mu\nu}\tilde{F}_{\rho\lambda} \right]. \quad (14)$$

The equation of motion for $B_{\mu\nu}$ which follows from this action is

$$\mathcal{D}_\lambda \tilde{H}^{\mu\nu\lambda} + \frac{m}{2}\epsilon^{\mu\nu\rho\lambda}(F_{\rho\lambda} - \tilde{F}_{\rho\lambda}) = 0, \quad (15)$$

and this is again solved by the ansatz $\tilde{H}^{\mu\nu\lambda} = -2m\epsilon^{\mu\nu\lambda\rho}\Phi_\rho$, this time without the requirement of flatness $\tilde{F}_{\rho\lambda} = 0$. The analysis proceeds as before, and we get the reduced action

$$S_{red} = \int d^4x \text{Tr} \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2m^2\Phi_\mu\Phi^\mu + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}E_{\mu\nu}\tilde{F}_{\rho\lambda} \right]. \quad (16)$$

In terms of path integrals, the action (5) corresponds to

$$\int [\mathcal{D}A][\mathcal{D}\tilde{A}][\mathcal{D}B]\delta(\tilde{F}_{\rho\lambda}) \exp(i \int \text{Tr}(-\frac{1}{12}\tilde{H}^2 - \frac{1}{4}F^2 + \frac{m}{2}B \wedge F)), \quad (17)$$

while the reduced action corresponds to

$$\int [\mathcal{D}A][\mathcal{D}\tilde{A}]\delta(\tilde{F}_{\rho\lambda}) \exp(i \int \text{Tr}(-\frac{1}{4}F^2 - 2m^2\Phi^2)). \quad (18)$$

In both the path integrals, the $\delta(F_{\rho\lambda})$ can be replaced by a term $\text{Tr}\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}E_{\mu\nu}\tilde{F}_{\rho\lambda}$ in the Lagrangian. In other words, the duality relation makes it possible to integrate out the B -field without getting involved in the subtleties in enforcing the flatness of \tilde{A} at the quantum level.

There are several things to be noticed about this duality, in particular about the action (12). If one were to formulate this system for an abelian gauge group one would find that the duality relation (8) was in fact the well known duality for the abelian model [5]. In other words, the model (5) is the non-abelian generalization of the mass generation mechanism if one were to start from the duality relation rather than from the action itself. Just as in the abelian case, the duality between Φ and the non-abelian two-form forbids any further

interaction terms of mass dimension four involving Φ . In particular a $(\Phi^\mu \Phi_\mu)^2$ or similar interaction cannot be added to the action without destroying the duality. Nor can one add any kinetic terms for Φ (that is, other than $(\mathcal{D}_{[\mu} \Phi_{\nu]})^2$, which can be rewritten as $(F_{\mu\nu})^2$).

Classically speaking, since \tilde{A} is a flat connection, it is possible to find a g such that $\tilde{A}_\mu = -\partial_\mu g g^{-1}$. Then the action (12) can be written as

$$\begin{aligned} S &= \int d^4x \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} (A_\mu + \partial_\mu g g^{-1})^2 \right) \\ &= \int d^4x \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} (D_\mu g g^{-1})^2 \right). \end{aligned} \quad (19)$$

This shows that, at least classically, the system described by (12) is in fact a gauged non-linear sigma model, which is known to be non-renormalizable. However it would be inappropriate to dismiss the action (12) (and by inference (5)) as non-renormalizable or non-unitary. In the case of (5), the duality relation (8) is highly non-local, which means that the longitudinal modes of massive gauge vectors in this model are related to the two-form through non-local relations. In the path integral quantization of the theory this implies a non-trivial Jacobian when a change of variables is made from $B_{\mu\nu}$ to Φ_μ . This transformation modifies the large momentum behaviour of the propagator and would introduce non-renormalizability. In canonical quantization the non-locality of the duality relation means that the amplitude of scattering longitudinal vectors off longitudinal vectors cannot be directly calculated from (5). In the case of (1) things are even more complicated as no duality exists, and longitudinal vectors cannot even be related to the antisymmetric tensor in a closed form. All attempts to prove or disprove tree-level unitarity of this model must therefore fail, and one has to look for alternative approaches, using BRST invariance for example [9,13], to quantize the theory in a self consistent manner.

There is yet another loophole which may protect the dual model (5) from non-unitarity and non-renormalizability. The relation $\tilde{A}_\mu = -\partial_\mu g g^{-1}$ is a non-local relation which contributes a non-trivial Jacobian when one makes a change of variables in the path integral³.

³I am indebted to A. Niemi for discussing and clarifying this point.

It is then by no means obvious whether the model of (12) is still identical to the non-linear sigma model as a quantum theory or if it escapes the fate of the latter through this loophole, which another model [12] has tried to exploit recently. In that model the field strength F in the interaction $B \wedge F$ is the field strength of the flat connection \tilde{A} , and Φ_μ , rather than a dynamical $B_{\mu\nu}$, is treated as a fundamental variable. The action proposed there is similar, but not identical, to the dualized action (12). The main difference lies in the presence of $(\Phi_\mu)^4$ interactions, and a symmetric kinetic term of the form $(D_{(\mu}\Phi_{\nu)})^2$ (the antisymmetric part can be rewritten as $(F_{\mu\nu})^2$). These two terms make sure that one cannot recover something like (5) by a duality transformation.

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